Sequence selection principles for functions

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Sequence Selection Principles

A.V. Arkhangel'skiĭ [1972]

properties $(\alpha_1) - (\alpha_4)$

For $i = 1, 2, 3, 4$, a topological space Y is (α_i)-space if for any sequence $\langle S_n : n \in \omega \rangle$ of sequences converging to a point $y \in Y$, there exists a sequence S converging to y such that:

(α_1) $S_n \setminus S$ is finite for all $n \in \omega$;

(α_2) $S_n \cap S$ is infinite for all $n \in \omega$;

(α_3) $S_n \cap S$ is infinite for infinitely many $n \in \omega$;

(α_4) $S_n \cap S \neq \emptyset$ for infinitely many $n \in \omega$.

D.H. Fremlin [1994]

equivalent conditions to an s_1-space

M. Scheepers [1997]

Sequence Selection Property SSP, Monotonic Sequence Selection Property MSSP

A topological space X has sequence selection property, if for any $x \in X$ and for any sequence $\langle S_n : n \in \omega \rangle$ of sequences converging to x there is a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n \to x$ and $x_n \in S_n$ for any $n \in \omega$.

All spaces are assumed to be Hausdorff and infinite.

Diagrams hold for perfectly normal space.

${}^{X}\mathbb{R}$	the space of all real-valued functions on X	(Tychonoff topology = t. of pointwise convergence)
$\mathrm{C}_p(X)$	the space of all continuous functions on X	(subspace topology)
$\mathcal{B}$	the space of all Borel functions on X	(subspace topology)
$\mathcal{U}$	the space of all upper semicontinuous functions on X with values in [0,1]	(subspace topology)

J. Gerlits and Zs. Nagy [1988], D.H. Fremlin [1994],
M. Scheepers [1998], [1999]

For a topological space X the following are equivalent.

(1) X is an s_1-space.

(2) $\mathrm{C}_p(X)$ has the sequence selection property.

(3) $\mathrm{C}_p(X)$ possesses (α_2).

(4) $\mathrm{C}_p(X)$ possesses (α_3).

(5) $\mathrm{C}_p(X)$ possesses (α_4).

perfectly normal space X

M. Scheepers [1998]

L. Bukovský and J. Haleš [2007]

M. Sakai [2007]

B. Tsaban and L. Zdomskyy [2012]

$C_p(X)$ possesses (α_1).

X is a QN-space. →

X is an $S_1(\Gamma_{\mathcal{B}}, \Gamma_{\mathcal{B}})$-space.

$\mathfrak{b}$-Sierpiński set

X is a σ-set

M. Scheepers [1999]

D.H. Fremlin [2003]

L. Bukovský and J. Haleš [2007]

$C_p(X)$ possesses (α_2).

X is a wQN-space. →

X is an $S_1(\Gamma_{\text{sh}}, \Gamma)$-space.

γ-set

X is perfectly meager

X is zero-dimensional

M. Scheepers [1999]

L. Bukovský, I. Reclaw and

M. Repický [2001]

L. Bukovský and J. Haleš [2003]

$C_p(X)$ possesses MSSP.

X is an mQN-space.

X has $\mathcal{U}_{fin}(\mathcal{O}_\omega, \Gamma)$.

compact set

X has count. Menger property

Convergence of $\langle f_n : n \in \omega \rangle$, $f_n, f : X \to \mathbb{R}$

Pointwise convergence P $f_n \xrightarrow{\mathrm{P}} f$

$$(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to |f_n(x) - f(x)| < \varepsilon)$$

Quasi-normal convergence Q $f_n \xrightarrow{\mathrm{Q}} f$
there exists $\langle \varepsilon_n : n \in \omega \rangle$ converging to 0 such that

$$(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to |f_n(x) - f(x)| < \varepsilon_n)$$

Discrete convergence D $f_n \xrightarrow{\mathrm{D}} f$

$$(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to f_n(x) = f(x))$$

Monotonic convergence M $f_n \xrightarrow{\mathrm{M}} f$

$f_n \xrightarrow{\mathrm{P}} f$ and $f_{n+1} \leq f_n$ for any $n \in \omega$

Properties $\mathrm{AB}(\mathcal{F},\mathcal{G})$ and $\mathrm{wAB}(\mathcal{F},\mathcal{G})$

$$
\begin{array}{cccccccccc}
\boldsymbol{f_{0,0}} & f_{0,1} & f_{0,2} & f_{0,3} & \cdots & f_{0,m} & \cdots & \xrightarrow{\mathrm{A}} & f_0 \\
f_{1,0} & f_{1,1} & f_{1,2} & f_{1,3} & \cdots & \boldsymbol{f_{1,m}} & \cdots & \xrightarrow{\mathrm{A}} & f_1 \\
f_{2,0} & f_{2,1} & f_{2,2} & \boldsymbol{f_{2,3}} & \cdots & f_{2,m} & \cdots & \xrightarrow{\mathrm{A}} & f_2 \\
 & \vdots & & \vdots & & \vdots & & & \\
f_{n,0} & f_{n,1} & f_{n,2} & f_{n,3} & \cdots & \boldsymbol{f_{n,m}} & \cdots & \xrightarrow{\mathrm{A}} & f_n \\
 & \vdots & & \vdots & & \vdots & & & \mathrm{A}\downarrow \\
 & & & & & & & \searrow^{\mathrm{B}} & f
\end{array}
$$

$\mathrm{A}, \mathrm{B} \in \{\mathrm{P}, \mathrm{Q}, \mathrm{D}\}$

pointwise P

quasi–normal Q

discrete D

$\mathcal{F}, \mathcal{G} \subseteq {}^X\mathbb{R}$

$0 \in \mathcal{F}, \mathcal{G}$

X has property $\mathrm{AB}(\mathcal{F},\mathcal{G})$ if for any $f_{n,m} \in \mathrm{C}_p(X)$, $f_n \in \mathcal{F}$, $f \in \mathcal{G}$ such that

$$f_{n,m} \xrightarrow{\mathrm{A}} f_n \text{ for every } n \in \omega \text{ and } f_n \xrightarrow{\mathrm{A}} f \text{ on } X$$

there exists an unbounded $\beta \in {}^\omega\omega$ such that $f_{n,\beta(n)} \xrightarrow{\mathrm{B}} f$ on X.

X satisfies principle $\mathrm{wAB}(\mathcal{F},\mathcal{G})$ if ... there exists an increasing $\alpha \in {}^\omega\omega$ and an unbounded $\beta \in {}^\omega\omega$ such that $f_{\alpha(n),\beta(n)} \xrightarrow{\mathrm{B}} f$ on X.

Sequence selection property $\mathrm{PP}(\{0\},\{0\})$

was considered by A.V. Arkhangel'skiĭ [1972] as property (α_2) for $\mathrm{C}_p(X)$ or M. Scheepers [1997] as sequence selection property for $\mathrm{C}_p(X)$.

Sequence selection property $\mathrm{wPP}(\{0\},\{0\})$

was considered by A.V. Arkhangel'skiĭ [1972] as property (α_4) for $\mathrm{C}_p(X)$.

Sequence selection property $\mathrm{DP}(\{0\},\{0\})$

was considered by L. Bukovský and J. Haleš [2007] as discrete sequence selection property.

Sequence selection properties $\mathrm{AB}({}^X\mathbb{R},{}^X\mathbb{R})$ **and** $\mathrm{AB}({}^X\mathbb{R},\{0\})$

were considered by L. Bukovský and J.Š. [2012] as ASB and ASB* selection principles.

$f_{n,m} \in \mathrm{C}_p(X), n, m \in \omega, \qquad f_{n,m} \xrightarrow{\mathrm{A}} f_n$ for every $n \in \omega$ and $f_n \xrightarrow{\mathrm{A}} f$ on X

f_n are F_σ-measurable functions on X

f is in second Baire class of functions on X

We will use $\mathcal{B}$ instead of ${}^X\mathbb{R}$.

$$
\begin{array}{ccccc}
 & & & & \mathrm{DD}(\mathcal{F},\mathcal{G}) \\
 & & & & \downarrow \\
\mathrm{PQ}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{QQ}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{DQ}(\mathcal{F},\mathcal{G}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{PP}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{QP}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{DP}(\mathcal{F},\mathcal{G})
\end{array}
$$

If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{G}_1 \subseteq \mathcal{G}_2$ then

$$\mathrm{AB}(\mathcal{F}_2,\mathcal{G}_2) \to \mathrm{AB}(\mathcal{F}_1,\mathcal{G}_1) \text{ and } \mathrm{wAB}(\mathcal{F}_2,\mathcal{G}_2) \to \mathrm{wAB}(\mathcal{F}_1,\mathcal{G}_1).$$

The family of sequence selection properties $\mathrm{AB}(\mathcal{F},\mathcal{G})$ and $\mathrm{wAB}(\mathcal{F},\mathcal{G})$ can be partially preordered by the relation

$$\mathcal{A} \leq \mathcal{D} \equiv \textbf{ZFC} \vdash \mathcal{D} \to \mathcal{A}.$$

Corresponding partially ordered set:

maximal elements are the equivalence classes of $\mathrm{PQ}(\mathcal{B},\mathcal{B})$ and $\mathrm{DD}(\mathcal{B},\mathcal{B})$

the smallest element is the equivalence class of $\mathrm{wDP}(\{0\},\{0\})$

maximal elements	the equivalence classes of $\mathrm{PQ}(\mathcal{B},\mathcal{B})$ and $\mathrm{DD}(\mathcal{B},\mathcal{B})$
the smallest element	the equivalence class of $\mathrm{wDP}(\{0\},\{0\})$

L. Bukovský – J.Š. [2012]

A perfectly normal space X has property $\mathrm{PQ}(\mathcal{B},\mathcal{B})$ if and only if X is a QN-space.
A perfectly normal space X has property $\mathrm{DD}(\mathcal{B},\mathcal{B})$ if and only if X is a QN-space.

Corollary L. Bukovský, I. Reclaw and M. Repický [1991]

Any $\mathfrak{b}$-Sierpiński set has all selection properties $\mathrm{AB}(\mathcal{F},\mathcal{G})$ and $\mathrm{wAB}(\mathcal{F},\mathcal{G})$.

L. Bukovský – J. Haleš [2007], J.Š. [∞]

A topological space X has property $\mathrm{wDP}(\{0\},\{0\})$ if and only if X is a wQN-space.

R. Laver [1976], A. Dow [1990], B. Tsaban and L. Zdomskyy [2012]

$$\mathrm{AB}(\mathcal{F},\mathcal{G}) \equiv \mathrm{MN}(\mathcal{Q},\mathcal{H}) \equiv \mathrm{wAB}(\mathcal{F},\mathcal{G}) \equiv \mathrm{wMN}(\mathcal{Q},\mathcal{H}) \text{ holds in Laver model.}$$

A.W. Miller and B. Tsaban [2010]

In Laver model, a perfectly normal space X has $\mathrm{AB}(\mathcal{F},\mathcal{G})$ if and only if $|X| < \mathfrak{b}$.

L. Bukovský, I. Reclaw and M. Repický [1991]

A topological space X is a QN-space (a wQN-space) if each sequence of continuous real-valued functions converging to zero on X is (has a subsequence) converging quasi-normally.

A set $X \subseteq \mathbb{R}$ is $\mathfrak{b}$-Sierpiński set if $|X| \geq \mathfrak{b}$ and $|A \cap X| < \mathfrak{b}$ for any Lebesgue measure zero set.

$$\mathrm{AB}(\{0\},\mathcal{G}) \equiv \mathrm{AB}(\{0\},\{0\}) \qquad \mathrm{wAB}(\{0\},\mathcal{G}) \equiv \mathrm{wAB}(\{0\},\{0\})$$

$\mathcal{G} \subseteq \mathcal{F}$, $\mathcal{G} \subseteq \mathrm{C}_p(X)$, $\mathcal{F}$ is closed under subtraction (L. Bukovský – J.Š. [2012])

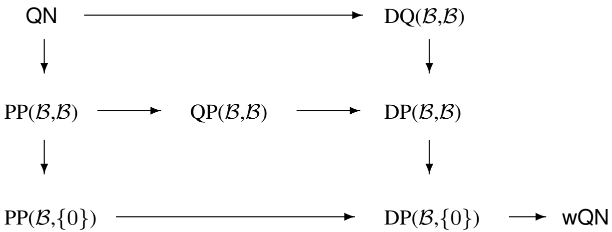
$$\mathrm{AB}(\mathcal{F},\mathcal{G}) \equiv \mathrm{AB}(\mathcal{F},\{0\}) \qquad \mathrm{wAB}(\mathcal{F},\mathcal{G}) \equiv \mathrm{wAB}(\mathcal{F},\{0\})$$

$$\mathrm{wAB}(\{0\},\{0\}) \equiv \mathrm{AB}(\{0\},\{0\})$$

$(\mathrm{A},\mathrm{B}) \neq (\mathrm{P},\mathrm{Q})$, $\mathcal{F} \subseteq \mathrm{C}_p(X)$

$$\mathrm{AB}(\mathcal{F},\mathcal{G}) \equiv \mathrm{AB}(\mathcal{F},\{0\}) \equiv \mathrm{AB}(\{0\},\{0\}) \equiv \mathrm{wAB}(\mathcal{F},\mathcal{G}) \equiv \mathrm{wAB}(\mathcal{F},\{0\})$$

$\mathcal{F}, \mathcal{G} \in \{\mathcal{B}, \mathrm{C}_p(X), \{0\}\}$



$$
\begin{array}{ccccccccc}
 & & & & \gamma\text{-set} & & & & \\
 & & & & \downarrow & & & & \\
\text{QN} & \longrightarrow & \mathrm{hS}_1(\Gamma,\Gamma) & \longrightarrow & \mathrm{S}_1(\Gamma,\Gamma) & \longrightarrow & \text{wQN} & \longrightarrow & \mathcal{U}_{fin}(\mathcal{O}_\omega,\Gamma) \\
 & & \downarrow & & & & & & \\
 & & \sigma\text{-set} & & & & & &
\end{array}
$$

J. Gerlits and Zs. Nagy [1982]

A topological space X is a γ-space if any open ω-cover of X contains γ-subcover.

M. Scheepers [1996]

A topological space X is an $\mathrm{S}_1(\Gamma,\Gamma)$-space if for every sequence $\langle \mathcal{A}_n : n \in \omega \rangle$ of open γ-covers of X there exist sets $U_n \in \mathcal{A}_n, n \in \omega$ such that $\{U_n;\ n \in \omega\}$ is a γ-cover.

A topological space X possesses $\mathcal{U}_{fin}(\mathcal{O}_\omega,\Gamma)$ if for any sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of countable open covers not containing a finite subcover there are finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n, n \in \omega$ such that $\{\bigcup \mathcal{V}_n;\ n \in \omega\}$ is a γ-cover.

For a property $\mathcal{A}$ of a topological space X we say that X is hereditarily $\mathcal{A}$-space, shortly h$\mathcal{A}$-space, or X possesses $\mathcal{A}$ hereditarily if any subset of X is an $\mathcal{A}$-space.

A topological space X is a σ-set if every F_σ subset of X is a G_δ set in X. ($\leq$1933)

A cover $\mathcal{A}$ of X is an ω-cover if for any finite subset F of X there is $A \in \mathcal{A}$ such that $F \subseteq A$.

An infinite cover $\mathcal{A}$ is a γ-cover if every $x \in X$ lies in all but finitely many members of $\mathcal{A}$.

$$
\begin{array}{ccccccccc}
 & & & & \gamma\text{-set} & & & & \\
 & & & & \downarrow & & & & \\
\text{QN} & \longrightarrow & \mathrm{hS}_1(\Gamma,\Gamma) & \longrightarrow & \mathrm{S}_1(\Gamma,\Gamma) & \longrightarrow & \text{wQN} & \longrightarrow & \mathcal{U}_{fin}(\mathcal{O}_\omega,\Gamma) \\
 & & \downarrow & & & & & & \\
 & & \sigma\text{-set} & & & & & &
\end{array}
$$

F. Galvin and A.W. Miller [1984], W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki [1996], M. Scheepers [1998], T. Orenshtein and B. Tsaban [2011]

If $\mathfrak{p} = \mathfrak{b}$ then there is a γ-set of reals of cardinality $\mathfrak{b}$ which is not a σ-set.

A topological space X is a γ-space if any open ω-cover of X contains γ-subcover.

A topological space X is an $\mathrm{S}_1(\Gamma,\Gamma)$-space if for every sequence $\langle \mathcal{A}_n : n \in \omega\rangle$ of open γ-covers of X there exist sets $U_n \in \mathcal{A}_n, n \in \omega$ such that $\{U_n;\ n \in \omega\}$ is a γ-cover.

A topological space X possesses $\mathcal{U}_{fin}(\mathcal{O}_\omega,\Gamma)$ if for any sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of countable open covers not containing a finite subcover there are finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n, n \in \omega$ such that $\{\bigcup \mathcal{V}_n;\ n \in \omega\}$ is a γ-cover.

For a property $\mathcal{A}$ of a topological space X we say that X is hereditarily $\mathcal{A}$-space, shortly h$\mathcal{A}$-space, or X possesses $\mathcal{A}$ hereditarily if any subset of X is an $\mathcal{A}$-space.

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An infinite cover $\mathcal{A}$ is a γ-cover if every $x \in X$ lies in all but finitely many members of $\mathcal{A}$.

Corollary

$\mathrm{Ind}(X) = 0$ for any normal space X having any of the selection properties $\mathrm{AB}(\mathcal{F},\mathcal{G})$ or $\mathrm{wAB}(\mathcal{F},\mathcal{G})$. A subset of metric separable space having any of the selection properties $\mathrm{AB}(\mathcal{F},\mathcal{G})$ or $\mathrm{wAB}(\mathcal{F},\mathcal{G})$ is perfectly meager.

J.Š. [∞]

A perfectly normal space X having $\mathrm{wDP}(\mathcal{U},\{0\})$ is an $\mathrm{S}_1(\Gamma,\Gamma)$-space.

L. Bukovský – J.Š. [2012]

If a perfectly normal topological space X has $\mathrm{wDP}(\mathcal{U},\mathcal{B})$ or $\mathrm{DP}(\mathcal{U},\{0\})$ then X is a σ-set.

Corollary

If a perfectly normal space X has $\mathrm{wDP}(\mathcal{U},\mathcal{B})$ or $\mathrm{DP}(\mathcal{U},\{0\})$ then X is hereditarily $\mathrm{S}_1(\Gamma,\Gamma)$-space.

J.Š. [∞]

If a perfectly normal topological space X has $\mathrm{wDP}(\mathcal{U},\{0\})$ then every open γ-cover of X is shrinkable.

J.Š. [∞]

Let X be a topological space. If X has $\mathrm{wDD}(\{0\},\{0\})$ or $\mathrm{PQ}(\mathrm{C}_p(X),\{0\})$ then X is a QN-space.

A cover $\mathcal{B}$ is said to be a refinement of $\mathcal{A}$ if for any $V \in \mathcal{B}$ there is $U \in \mathcal{A}$ such that $V \subseteq U$.

A γ-cover $\mathcal{A}$ is shrinkable if there exists a closed γ-cover $\mathcal{B}$ which is a refinement of $\mathcal{A}$.

Surprising result

J.Š. [∞]

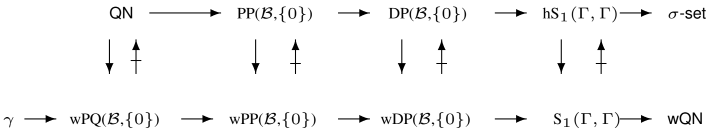
Any γ-set has property $\mathrm{wPQ}(\mathcal{B},\{0\})$.

$$\mathrm{AB}(\{0\},\{0\}) \equiv \mathrm{wAB}(\{0\},\{0\})$$

$$\mathrm{AB}(\mathrm{C}_p(X),\mathcal{B}) \equiv \mathrm{wAB}(\mathrm{C}_p(X),\mathcal{B})$$

for $(\mathrm{A},\mathrm{B}) \neq (\mathrm{P},\mathrm{Q}):\ \mathrm{AB}(\mathrm{C}_p(X),\{0\}) \equiv \mathrm{wAB}(\mathrm{C}_p(X),\{0\})$

$\mathfrak{p} = \mathfrak{b}$



Distinguishing

$\mathfrak{p} = \mathfrak{b}$, $(\mathrm{A}, \mathrm{B}) \neq (\mathrm{P}, \mathrm{Q})$, $\mathrm{B} \neq \mathrm{D}$

$$\mathrm{wAB}(\mathcal{B},\mathcal{B})$$

$$\mathrm{QN} \equiv \mathrm{PQ}(\mathrm{C}_p(X),\{0\}) \longrightarrow \mathrm{AB}(\mathcal{B},\mathcal{B}) \qquad \mathrm{hS}_1(\Gamma,\Gamma) \longrightarrow \sigma\text{-set}$$

$$\mathrm{AB}(\mathcal{B},\{0\})$$

$$\downarrow \qquad \not\uparrow$$

$$\mathrm{wQN} \equiv \mathrm{wPQ}(\mathrm{C}_p(X),\{0\}) \equiv \mathrm{AB}(\mathrm{C}_p(X),\{0\}) \equiv \mathrm{AB}(\{0\},\{0\}) \equiv \mathrm{PQ}(\{0\},\{0\})$$

Miller model, $(\mathrm{A}, \mathrm{B}) \neq (\mathrm{P}, \mathrm{Q})$**,** $\mathrm{B} \neq \mathrm{D}$

A.W. Miller [1979]

$$\mathrm{QN} \equiv \mathrm{PQ}(\mathrm{C}_p(X),\{0\}) \equiv \mathrm{AB}(\mathcal{B},\mathcal{B}) \equiv \mathrm{wAB}(\mathcal{B},\mathcal{B}) \equiv \mathrm{AB}(\mathcal{B},\{0\}) \equiv \mathrm{hS}_1(\Gamma,\Gamma) \equiv \sigma\text{-set}$$

$$\downarrow \qquad \not\uparrow$$

$$\mathrm{wQN} \equiv \mathrm{wPQ}(\mathrm{C}_p(X),\{0\}) \equiv \mathrm{AB}(\mathrm{C}_p(X),\{0\}) \equiv \mathrm{AB}(\{0\},\{0\}) \equiv \mathrm{PQ}(\{0\},\{0\})$$

perfecly normal space X

X is a QN-space	X has PQ($\mathcal{B}$,$\mathcal{B}$)	L. Bukovský - J.Š. [2012]
X is hereditarily $S_1(\Gamma, \Gamma)$-space	X has PP($\mathcal{U}$,$\{0\}$)	J.Š. [∞]
X is an $S_1(\Gamma, \Gamma)$-space and every open γ-cover of X is shrinkable	X has wPQ($\mathcal{U}$,$\{0\}$)	J.Š. [∞]
X is a wQN-space	X has PP($\{0\}$,$\{0\}$)	M. Scheepers [1999], D.H. Fremlin [2003]

J. Haleš [2005], H. Ohta and M. Sakai [2009], L. Bukovský and J.Š. [2012]

Properties $\mathrm{ABC}(\mathcal{F},\mathcal{G})$ and $\mathrm{wABC}(\mathcal{F},\mathcal{G})$

$$
\begin{array}{cccccccccc}
\boldsymbol{f_{0,0}} & f_{0,1} & f_{0,2} & f_{0,3} & \cdots & f_{0,m} & \cdots & \xrightarrow{\mathrm{A}} & f_0 \\
f_{1,0} & f_{1,1} & f_{1,2} & f_{1,3} & \cdots & \boldsymbol{f_{1,m}} & \cdots & \xrightarrow{\mathrm{A}} & f_1 \\
f_{2,0} & f_{2,1} & f_{2,2} & \boldsymbol{f_{2,3}} & \cdots & f_{2,m} & \cdots & \xrightarrow{\mathrm{A}} & f_2 \\
 & \vdots & & \vdots & & \vdots & & & \\
f_{n,0} & f_{n,1} & f_{n,2} & f_{n,3} & \cdots & \boldsymbol{f_{n,m}} & \cdots & \xrightarrow{\mathrm{A}} & f_n \\
 & \vdots & & \vdots & & \vdots & & & \mathrm{B}\downarrow \\
 & & & & & & & \searrow^{\mathrm{C}} & f
\end{array}
$$

$\mathrm{A}, \mathrm{B}, \mathrm{C} \in \{\mathrm{P}, \mathrm{Q}, \mathrm{D}, \mathrm{M}\}$

pointwise P

quasi–normal Q

discrete D

monotonic M

$\mathcal{F}, \mathcal{G} \subseteq {}^X\mathbb{R}$

$0 \in \mathcal{F}, \mathcal{G}$

X has property $\mathrm{ABC}(\mathcal{F},\mathcal{G})$ if for any $f_{n,m} \in \mathrm{C}_p(X)$, $f_n \in \mathcal{F}$, $f \in \mathcal{G}$ such that

$$f_{n,m} \xrightarrow{\mathrm{A}} f_n \text{ for every } n \in \omega \text{ and } f_n \xrightarrow{\mathrm{B}} f \text{ on } X$$

there exists an unbounded $\beta \in {}^\omega\omega$ such that $f_{n,\beta(n)} \xrightarrow{\mathrm{C}} f$ on X.

X satisfies principle $\mathrm{wABC}(\mathcal{A},\mathcal{B})$ if ... there exists an increasing $\alpha \in {}^\omega\omega$ and an unbounded $\beta \in {}^\omega\omega$ such that $f_{\alpha(n),\beta(n)} \xrightarrow{\mathrm{C}} f$ on X.

perfecly normal space X

X is a QN-space	X has PQ($\mathcal{B}$,$\mathcal{B}$)	L. Bukovský - J.Š. [2012]
X is hereditarily $\mathrm{S}_1(\Gamma, \Gamma)$-space	X has PP($\mathcal{U}$,$\{0\}$)	J.Š. [∞]
X is an $\mathrm{S}_1(\Gamma, \Gamma)$-space and every open γ-cover of X is shrinkable	X has wPQ($\mathcal{U}$,$\{0\}$)	J.Š. [∞]
X is a wQN-space	X has PP($\{0\}$,$\{0\}$)	M. Scheepers [1999], D.H. Fremlin [2003]
X possesses Hurewicz property hereditarily	X has MPP($\mathcal{B}$,$\{0\}$)	J.Š. [∞]
X possesses Hurewicz property and every open γ-cover of X is shrinkable	X has wMPP($\mathcal{B}$,$\{0\}$)	J.Š. [∞]
X possesses USC_m and Hurewicz property	X has MMP($\mathcal{B}$,$\{0\}$)	J.Š. [∞]
X possesses Hurewicz property	X has MMP($\{0\}$,$\{0\}$)	M. Scheepers [1997]

J. Haleš [2005], H. Ohta and M. Sakai [2009], T. Orenshtein and B. Tsaban [2011], B. Tsaban and L. Zdomskyy [2012], L. Bukovský and J.Š. [2012], M. Scheepers [1997]

Hurewicz property = property $\mathcal{U}_{fin}(\mathcal{O}_\omega, \Gamma)$

Property USC_m introduced and investigated by H. Ohta and M. Sakai [2009].

M. Scheepers [1999],D.H. Fremlin [2003] $\quad \mathrm{B} \neq \mathrm{D}$

top.

X satisies $\mathrm{AB}(\{0\},\{0\})$ if and only if X is a wQN-space.

sp. X

J.Š. [∞] $\quad X$ is a wQN-space if and only if X has $\mathrm{wPQ}(\mathrm{C}_p(X),\{0\})$.

L. Bukovský – J.Š. [2012]

p.n.

sp. X

X has $\mathrm{DD}(\mathcal{F},\mathcal{G})$ if and only if X is a QN-space.

X has $\mathrm{wDD}(\mathcal{F},\mathcal{G})$ if and only if X is a QN-space.

L. Bukovský – J.Š. [2012], J.Š. [∞]

$\mathrm{C}_p(X) \subseteq \mathcal{F}$

X has $\mathrm{PQ}(\mathcal{F},\mathcal{G})$ if and only if X is a QN-space.

X has $\mathrm{wPQ}(\mathcal{F},\mathcal{B})$ if and only if X is a QN-space.

L. Bukovský – J.Š. [2012]

X is a QN-space if and only if X has $\mathrm{wQQ}(\mathcal{B},\mathcal{B})$ if and only if X has $\mathrm{QQ}(\mathcal{B},\mathcal{B})$.

L. Bukovský – J.Š. [∞]

$$\mathrm{QQ}(\mathcal{B},\{0\}) \equiv \mathrm{DQ}(\mathcal{B},\{0\}) \equiv \mathrm{QP}(\mathcal{B},\{0\}) \equiv \mathrm{DP}(\mathcal{B},\{0\})$$

$$\mathrm{wQQ}(\mathcal{B},\{0\}) \equiv \mathrm{wDQ}(\mathcal{B},\{0\}) \equiv \mathrm{wQP}(\mathcal{B},\{0\}) \equiv \mathrm{wDP}(\mathcal{B},\{0\})$$

J.Š. [∞] $\quad \mathrm{B} \neq \mathrm{D}$

If $(\mathrm{A},\mathrm{B}) \neq (\mathrm{P},\mathrm{Q})$ then X has $\mathrm{AB}(\mathcal{U},\{0\})$ if and only if X is hereditarily $\mathrm{S}_1(\Gamma,\Gamma)$-space.

X has $\mathrm{wAB}(\mathcal{U},\{0\})$ if and only if X is an $\mathrm{S}_1(\Gamma,\Gamma)$-space and every open γ-cover of X is shrinkable.

Application

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Let X be a topological space.

D.H. Fremlin [1994], M. Scheepers [1999]

X has PP{0}{0} if and only if $\mathrm{scl}_{\omega_1}(A, \mathrm{C}_p(X)) = \mathrm{scl}_1(A, \mathrm{C}_p(X))$ for every $A \subseteq \mathrm{C}_p(X)$.

J.Š. [∞]

X has wPP($\mathcal{B}$,$\mathcal{B}$) if and only if $\mathrm{scl}_{\omega_1}(A, {}^X\mathbb{R}) = \mathrm{scl}_1(A, {}^X\mathbb{R})$ for every $A \subseteq \mathrm{C}_p(X)$.

X has wPP($\mathcal{B}$,$\{0\}$) if and only if $\mathrm{scl}_2(A, {}^X\mathbb{R}) \cap \mathrm{C}_p(X) = \mathrm{scl}_1(A, {}^X\mathbb{R}) \cap \mathrm{C}_p(X)$ for any $A \subseteq \mathrm{C}_p(X)$.

$$A \subseteq Y: \quad \mathrm{scl}(A, Y) = \{y \in Y;\ (\exists \{y_n\}_{n=0}^{\infty} \in {}^{\omega}A)\ y_n \to y\}$$

$$\mathrm{scl}_0(A, Y) = A, \quad \mathrm{scl}_\alpha(A, Y) = \mathrm{scl}\left(\bigcup_{\beta<\alpha} \mathrm{scl}_\beta(A, Y), Y\right), \alpha > 0$$

T. Orenshtein [2009]
X possesses property $\binom{\mathsf{S}_0'}{\Gamma_0}$ if for any set $A \subseteq \mathrm{C}_p(X) \setminus \{0\}$ with $0 \in \mathrm{scl}_{\omega_1}(A, {}^X\mathbb{R})$ there is a sequence $\langle f_n : n \in \omega \rangle$ of functions from A such that $f_n \to 0$.

J.Š. [∞]
The statements

“$\mathrm{scl}_{\omega_1}(A, {}^X\mathbb{R}) = \mathrm{scl}_1(A, {}^X\mathbb{R})$ for every $A \subseteq \mathrm{C}_p(X)$ for any perfectly normal $\mathrm{S}_1(\Gamma, \Gamma)$-space X”,
“$\mathrm{scl}_{\omega_1}(A, {}^X\mathbb{R}) = \mathrm{scl}_1(A, {}^X\mathbb{R})$ for every $A \subseteq \mathrm{C}_p(X)$ for any perfectly normal space X possessing $\binom{\mathsf{S}_0'}{\Gamma_0}$”

are undecidable in **ZFC**. The theory

ZFC+“any perfectly normal $\mathrm{S}_1(\Gamma, \Gamma)$-space possesses $\binom{\mathsf{S}_0'}{\Gamma_0}$”

is consistent with **ZFC**.

Solutions and partial solution to Problems 6.0.15, 6.0.16 and 6.0.17 of T. Orenshtein [2009].

$A \subseteq Y$: $\mathrm{scl}(A, Y) = \{y \in Y;\ (\exists \{y_n\}_{n=0}^{\infty} \in {}^{\omega}A)\ y_n \to y\}$

$\mathrm{scl}_0(A, Y) = A, \mathrm{scl}_\alpha(A, Y) = \mathrm{scl}\left(\bigcup_{\beta<\alpha} \mathrm{scl}_\beta(A, Y), Y\right), \alpha > 0$

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B. Tsaban – L. Zdomskyy [2012], announcement 2006

If X is a perfectly normal topological space, then X is a QN-space if and only if any Borel measurable function $f : X \longrightarrow {}^{\omega}\omega$ is eventually bounded.

L. Bukovský – J.Š. [∞]

A topological space X possesses the JR-property if every Δ^0_2 measurable real function defined on X is a discrete limit of a sequence of continuous functions.

J.E. Jayne and C.A. Rogers 1982 — Any analytic subset of a Polish space has the JR-property.

L. Bukovský, I. Reclaw and M. Repický [2001]

If X is a perfectly normal topological space, then X is a QN-space with the JR-property if and only if any Borel measurable function $f : X \longrightarrow {}^{\omega}\omega$ is eventually bounded.

L. Bukovský – J.Š. [2012]

Any QN-space has property $\mathrm{QQ}(\mathcal{B},\mathcal{B})$.

L. Bukovský – J.Š. [2012]

If a perfectly normal space X has $\mathrm{QQ}(\mathcal{B},\mathcal{B})$, then X has the JR-property.

Application

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I. Reclaw [1997], L. Bukovský, I. Reclaw and M. Repický [2001], J. Haleš [2005]

A perfectly normal QN-space is a σ-space.

L. Bukovský – J.Š. [2012]

Any QN-space has property $\mathrm{QQ}(\mathcal{B},\mathcal{B})$.

L. Bukovský – J.Š. [2012]

If a perfectly normal topological space X has $\mathrm{wDP}(\mathcal{U},\mathcal{B})$ then X is a σ-set.

Архангельский А.В. (Arkhangel'skiĭ A.V.), *Спектр частот топологического пространства и классификация пространств*, ДАН СССР, **206**:2 (1972), 265–268. English translation *The frequency spectrum of a topological space and the classification of spaces,* Soviet Math. Dokl. **13** (1972), 1185–1189.

Bukovský L. and Haleš J., *On Hurewicz properties,* Topology Appl. **132** (2003), 71–79.

Bukovský L. and Haleš J., QN-*spaces,* wQN-*spaces and covering properties,* Topology Appl. **154** (2007), 848–858.

Bukovský L., Rec law I. and Repický M., *Spaces not distinguishing pointwise and quasinormal convergence of real functions,* Topology Appl. **41** (1991), 25–40.

Bukovský L., Rec law I. and Repický M., *Spaces not distinguishing convergences of real–valued functions,* Topology Appl. **112** (2001), 13–40.

Bukovský L. and Šupina J., *Sequence selection principles for quasi-normal convergence,* Topology Appl. **159** (2012), 283–289.

Bukovský L. and Šupina J., *Modifications of sequence selection principles,* to appear.

Dow A., *Two classes of Fréchet-Urysohn spaces,* Proc. Amer. Math. Soc. **108** (1990), 241–247.

Fremlin D.H., *Sequential convergence in* $\mathrm{C}_p(X)$*,* Comment. Math. Univ. Carolin. **35** (1994), 371–382.

Fremlin D.H., SSP *and* wQN*,* Notes of 14.01.2003, http://www.essex.ac.uk/maths/people/fremlin/preprints.htm.

Galvin F. and Miller A.W., γ*-sets and other singular sets of real numbers,* Topology Appl. **17** (1984), 145–155.

Gerlits J. and Nagy Zs., *Some properties of C(X), I,* Topology Appl. **14** (1982), 151–161.

Gerlits J. and Nagy Zs., *On Frechet spaces,* Rend. Circ. Mat. Palermo (2), Suppl. No. **18** (1988), 51–71.

Haleš J., *On Scheepers' conjecture,* Acta Univ. Carolinae Math. Phys. **46** (2005), 27–31.

Jayne J.E. and Rogers C.A., *First level Borel functions and isomorphisms,* J. Math. pures et appl. **61** (1982), 177–205.

Just W., Miller A.W., Scheepers M. and Szeptycki P.J., *Combinatorics of open covers* II*,* Topology Appl. **73** (1996), 241–266.

Laver R., *On the consistency of Borel's conjecture,* Acta Math. **137** (1976), 151–169.

Miller A.W. and Tsaban B., *Point-cofinite covers in the Laver model,* Proc. Amer. Math. Soc. **138** (2010), 3313–3321.

Miller A.W., *On the length of Borel hierarchies,* Ann. Math. Logic **16** (1979), 233–267.

Ohta H. and Sakai M., *Sequences of semicontinuous functions accompanying continuous functions*, Topology Appl. **156** (2009), 2683-2906.

Orenshtein T., *Global topological properties and convergence of real functions,* M.Sc. Thesis, Weizmann Institute of Science, 2009.

Orenshtein T. and Tsaban B., *Linear σ-additivity and some applications,* Trans. Amer. Math. Soc. **363** (2011), 3621–3637.

Reclaw I., *Metric spaces not distinguishing pointwise and quasinormal convergence of real functions,* Bull. Acad. Polon. Sci. **45** (1997), 287–289.

- Sakai M., *The sequence selection properties of* $C_p(X)$, Topology Appl. **154** (2007), 552–560.
- Scheepers M., *Combinatorics of open covers I: Ramsey theory,* Topology Appl. **69** (1996), 31–62.
- Scheepers M., *A sequential property of* $C_p(X)$ *and a covering property of Hurewicz,* Proc. Amer. Math. Soc. **125** (1997), 2789–2795.
- Scheepers M., $C_p(X)$ *and Archangel'skiĭ's* α_i*-spaces,* Topology Appl. **45** (1998), 265–275.
- Scheepers M., *Sequential convergence in* $C_p(X)$ *and a covering property,* East-West J. of Mathematics **1** (1999), 207–214.
- Šupina J., *A remark on sequence selection property,* to appear.
- Tsaban B. and Zdomskyy L., *Hereditary Hurewicz spaces and Arhangel'skiĭ sheaf amalgamations,* J. Eur. Math. Soc. (JEMS) **14** (2012), 353–372.

Thanks for Your attention!